Isometries; Conformal Maps

Definition A diffeomorphism $\varphi : S \to \bar{S}$ is an isometry if

$$\langle w_1, w_2\rangle_p = \langle d\varphi_p(w_1), d\varphi_p(w_2)\rangle_{\varphi(p)} \quad \forall\, p \in S,\ \forall\, w_1,\, w_2 \in T_pS.$$

The surfaces S and $\bar{S}$ are then said to be isometric.

In other words, a diffeomorphism φ is an isometry if the differential $d\varphi$ preserves the inner product. It follows that, $d\varphi_p : T_pS \to T_{\varphi(p)}\bar{S}$ being an isometry,

$$I_p(w) = \langle w, w\rangle_p = \langle d\varphi(w), d\varphi(w)\rangle_{\varphi(p)} = I_{\varphi(p)}(d\varphi_p(w)) \quad \forall\, w \in T_pS,$$

i.e. the diffeomorphism $\varphi : S \to \bar{S}$ prserves the first fundamental form.

Conversely, if the diffeomorphism $\varphi : S \to \bar{S}$ prserves the first fundamental form, then

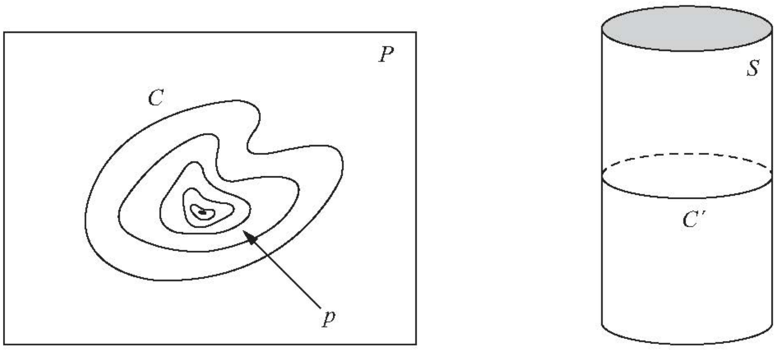
$$\begin{aligned} 2\langle w_1, w_2\rangle &= I_p(w_1 + w_2) - I_p(w_1) - I_p(w_2) \quad \forall\, w_1, w_2 \in T_pS \\ &= I_{\varphi(p)}(d\varphi_p(w_1 + w_2)) - I_{\varphi(p)}(d\varphi_p(w_1)) - I_{\varphi(p)}(d\varphi_p(w_2)) \\ &= \langle d\varphi(w_1), d\varphi(w_2)\rangle, \end{aligned}$$

and φ is, therefore, an isometry.

Definition A map $\varphi : V \to \bar{S}$ of a neighborhood $V \subset S$ of $p \in S$ is a local isometry at p if there exists a neighborhood $\bar{V} \subset \bar{S}$ of $\varphi(p) \in \bar{S}$ such that $\varphi : V \to \bar{V}$ is an isometry. If there exists a local isometry into $\bar{S}$ at every $p \in S$, the surface S is said to be locally isometric to $\bar{S}$.

It is clear that if $\varphi : S \to \bar{S}$ is a diffeomorphism and a local isometry for every $p \in S$, then φ is an isometry (globally).

However, a local isometry is not necessary an isometry globally, e.g. the xy-plane $P = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$ and the cylinder $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ are locally isometric, but they are not homeomorphic, so P and S are not diffeomorphic or isometric globally.



$C \subset P$ can be shrunk continuously into p without leaving P. The same does not hold for $C' \subset S$.

Since any simple closed curve $C \subset P$ in the plane P can be shrunk (deformed) continuously into a point without leaving the plane P, and this topological property in P is preserved by a homeomorphism $\varphi : P \to \varphi(P)$.

Note that a parallel C', e.g. $C' = \{(\cos u, \sin u, 0) \mid u \in [0, 2\pi]\} \subset S$, of the cylinder S does not have that property while the corresponding unit circle $C = \{(x, y, 0) \mid x^2 + y^2 = 1\}$ in P can be shrunk continuously into a point without leaving the plane P, so P and S are not homeomorphic.

Proposition Suppose that there exist parametrizations $X : U \to S$ and $\bar{X} : U \to \mathbb{S}$ such that $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$ in U. Then the map $\varphi = \bar{X} \circ X^{-1} : X(U) \to \bar{S}$ is a local isometry.

Proof Let $p \in X(U)$ and $w \in T_pS$. Then w is tangent to a curve $X(\alpha(t))$ at $t = 0$, where $\alpha(t) = (u(t), v(t))$ is a curve in U; thus, w may be written $(t = 0)$

$$w = X_u u' + X_v v'.$$

By definition, the vector $d\varphi_p(w)$ is the tangent vector to the curve $\bar{X} \circ X^{-1} \circ X(\alpha(t)) = \bar{X}(\alpha(t))$ at $t = 0$. Thus,

$$d\varphi_p(w) = \bar{X}_u u' + \bar{X}_v v'.$$

Since

$$\begin{aligned} I_p(w) &= E(u')^2 + 2Fu'v' + G(v')^2, \\ I_{\varphi(p)}(d\varphi_p(w)) &= \bar{E}(u')^2 + 2\bar{F}u'v' + \bar{G}(v')^2, \end{aligned}$$

and the assumption $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$ in U, we conclude that $I_p(w) = I_{\varphi(p)}(d\varphi_p(w))$ for all $p \in X(U)$ and all $w \in T_pS$; hence, φ is a local isometry.

Definition A diffeomorphism $\varphi : S \to \bar{S}$ is called a conformal map if

$$\langle d\varphi_p(w_1), d\varphi_p(w_2)\rangle = \lambda^2(p)\,\langle w_1, w_2\rangle \overset{w_1 = w_2}{\Longrightarrow} |d\varphi_p(w_1)|^2 = \lambda^2(p)\,|w_1|^2 \quad \forall\, p \in S, \forall\, w_1, w_2 \in T_pS,$$

where λ^2 is a nowhere-zero differentiable function on S; the surface S and $\bar{S}$ are then said to be conformal. A map $\varphi : V \to \bar{S}$ of a neighborhood $V \subset S$ of $p \in S$ is a local conformal map at p if there exists a neighborhood $\bar{V} \subset \bar{S}$ of $\varphi(p) \in \bar{S}$ such that $\varphi : V \to \bar{V}$ is a conformal map. If there exists a local conformal map into $\bar{S}$ at every $p \in S$, the surface S is said to be locally conformal to $\bar{S}$.

The geometric meaning of the above definition is that the angles (but not necessarily the lengths) are preserved by conformal maps. In fact, let $\alpha : I \to S$ and $\beta : I \to S$ be two curves in S which intersect at, say, $t = 0$. Their angle θ at $t = 0$ is given by

$$\cos\theta = \frac{\langle \alpha', \beta'\rangle}{|\alpha'|\,|\beta'|}, \quad 0 \le \theta \le \pi.$$

A conformal map $\varphi : S \to \bar{S}$ maps these curves into $\varphi \circ \alpha : I \to \bar{S}$, $\varphi \circ \beta : I \to \bar{S}$, which intersect when $t = 0$, making an angle $\bar{\theta}$ given by

$$\cos\bar{\theta} = \frac{\langle d\varphi(\alpha'), d\varphi(\beta')\rangle}{|d\varphi(\alpha')|\,|d\varphi(\beta')|} = \frac{\lambda^2\langle \alpha', \beta'\rangle}{\lambda^2|\alpha'|\,|\beta'|} = \cos\theta.$$

Proposition Suppose that there exist parametrizations $X : U \to S$ and $\bar{X} : U \to \bar{S}$ such that $E = \lambda^2\bar{E}$, $F = \lambda^2\bar{F}$, $G = \lambda^2\bar{G}$ in U, where λ^2 is a nowhere-zero differentiable function in U. Then the map $\varphi = \bar{X} \circ X^{-1} : X(U) \to \bar{S}$ is a local conformal map.

Example For $a > 0$, let

$$\begin{aligned} X(u, v) &= (a\cosh v\cos u, a\cosh v\sin u, av), \quad (u, v) \in U = \{0 < u < 2\pi,\ -\infty < v < \infty\} \\ \bar{X}(\bar{u}, \bar{v}) &= (\bar{v}\cos\bar{u}, \bar{v}\sin\bar{u}, a\bar{u}), \quad (\bar{u}, \bar{v}) \in U = \{0 < \bar{u} < 2\pi,\ -\infty < \bar{v} < \infty\} \end{aligned}$$

be parametrizations of the catenoid S and the helicoid $\bar{S}$, rspectively. Then the coefficients of the first fundamental forms are

$$E = a^2 \cosh^2 v, \quad F = 0, \quad G = a^2(1 + \sinh^2 v) = a^2 \cosh^2 v \quad \forall\,(u, v) \in U,$$
$$\bar{E} = a^2 + \bar{v}^2, \quad \bar{F} = 0, \quad \bar{G} = 1 \quad \forall\,(\bar{u}, \bar{v}) \in U.$$

Let us make the following change of parameters

$$\bar{u} = u, \quad \bar{v} = a \sinh v, \quad \forall\,(u, v) \in U,$$

which is possible since the map is clearly one-to-one, and the Jacobian

$$\frac{\partial(\bar{u}, \bar{v})}{\partial(u, v)} = a \cosh v \neq 0 \quad \forall\,(u, v) \in U.$$

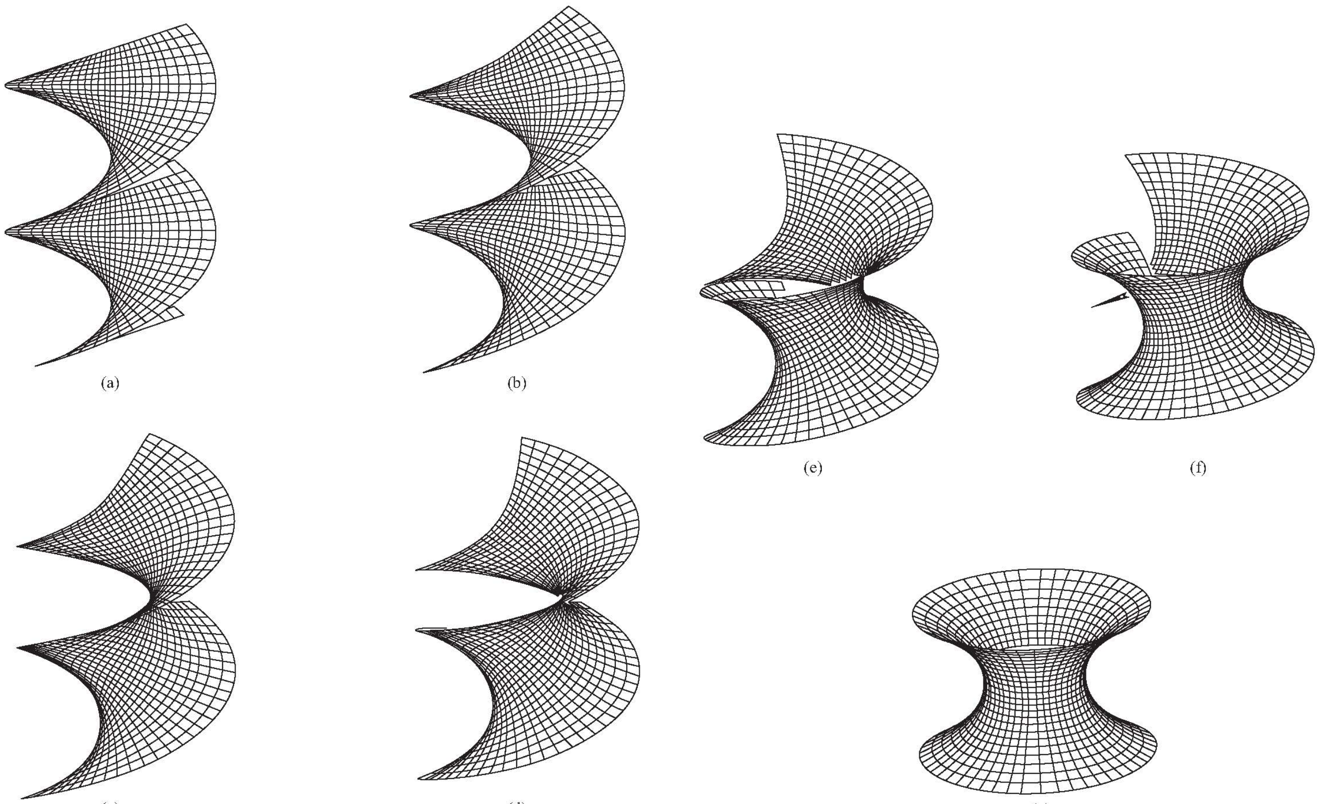
Thus,

$$\bar{X}(u, v) = (a \sinh v \cos u, a \sinh v \sin u, au), \quad (u, v) \in U,$$

is a new parametrization of the helicoid with

$$E = a^2 \cosh^2 v, \quad F = 0, \quad G = a^2 \cosh^2 v \quad \forall\,(u, v) \in U.$$

We conclude that the catenoid and the helicoid are locally isometric.



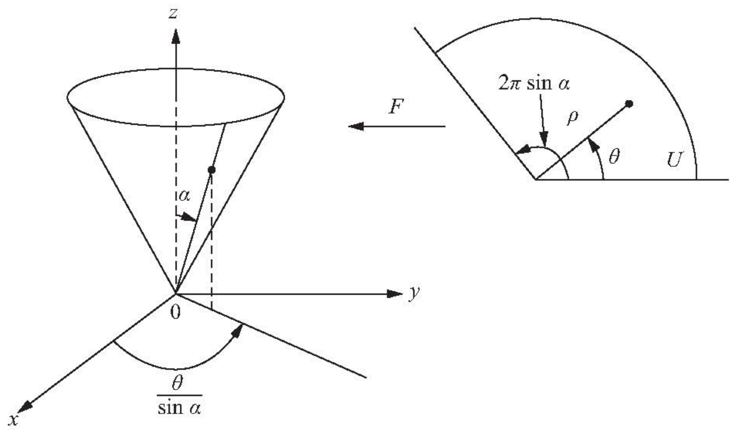
(a) (b) (e) (f) (c) (d) (g)

Example Let S be the one-sheeted cone (minus the vertex)

$$z = k\sqrt{x^2 + y^2}, \quad k > 0, \ (x, y) \neq (0, 0),$$

and let $U \subset \mathbb{R}^2$ be the open set given in polar coordinates (ρ, θ) by

$$0 < \rho < \infty, \quad 0 < \theta < 2\pi \sin \alpha,$$



where 2α $(0 < 2\alpha < \pi)$ is the angle at the vertex of the cone (i.e., where $\cot\alpha = k$), and let $F : U \to S \subset \mathbb{R}^3$ be the map

$$F(\rho,\theta) = \left(\rho\sin\alpha\cos\left(\frac{\theta}{\sin\alpha}\right), \rho\sin\alpha\sin\left(\frac{\theta}{\sin\alpha}\right), \rho\cos\alpha\right).$$

Then

- $F(U) \subset S$, since
$$k\sqrt{x^2+y^2} = \cot\alpha\,\sqrt{\rho^2\sin^2\alpha} = \rho\cos\alpha = z,$$
- $F : U \to S \setminus \{(\rho\sin\alpha, 0, \rho\cos\alpha) \mid 0 < \rho < \infty\}$ is a diffeomorphism from U onto the cone minus a generator $\theta = 0$, since F and dF are one-to-one in U,

and thus $F(\rho,\theta)$ is a parametrization of S with the coefficients of the first fundamental form being

$$E = \langle F_\rho, F_\rho\rangle = 1, \quad F = \langle F_\rho, F_\theta\rangle = 0, \quad G = \langle F_\theta, F_\theta\rangle = \rho^2,$$

Also since U may be viewed as a regular surface parametrized by

$$\bar{X}(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, 0) \in \mathbb{R}^3, \quad 0 < \rho < \infty, \quad 0 < \theta < 2\pi\sin\alpha,$$

with the coefficients of the first fundamental form of U in this parametrization being

$$\bar{E} = \langle \bar{X}_\rho, \bar{X}_\rho\rangle = 1 = E, \quad \bar{F} = \langle \bar{X}_\rho, \bar{X}_\theta\rangle = 0 = F, \quad \bar{G} = \langle \bar{X}_\theta, \bar{X}_\theta\rangle = \rho^2 = G,$$

$F : U \to S$ is a local isometry.

The most important property of conformal maps is given by the following theorem, which we shall not prove.

Theorem Any two regular surfaces are locally conformal.

The proof is based on the possibility of parametrizing a neighborhood of any point of a regular surface in such a way that the coefficients of the first fundamental form are

$$E = \lambda^2(u,v), \quad F = 0, \quad G = \lambda^2(u,v).$$

Such a coordinate system is called isothermal. Once the existence of an isothermal coordinate system of a regular surface S is assumed, S is clearly locally conformal to a plane, and by composition locally conformal to any other surface.

The Gauss Theorem and the Equations of Compatibility

Let $X : U \subset \mathbb{R}^2 \to S$ be a parametrization in the orientation of S. At each $p \in X(U)$, since $X_u,\ X_v,\ N \in \mathbb{R}^3$ are linearly independent, we may express vectors $X_{uu},\ X_{uv},\ X_{vu},\ X_{vv},\ N_u,\ N_v \in \mathbb{R}^3$ in the basis $\{X_u,\ X_v,\ N\}$ and obtain

$$\begin{aligned}
X_{uu} &= \Gamma^1_{11}X_u + \Gamma^2_{11}X_v + eN \\
X_{uv} &= \Gamma^1_{12}X_u + \Gamma^2_{12}X_v + fN \\
X_{vu} &= \Gamma^1_{21}X_u + \Gamma^2_{21}X_v + fN \\
X_{vv} &= \Gamma^1_{22}X_u + \Gamma^2_{22}X_v + gN \\
N_u &= a_{11}X_u + a_{21}X_v \\
N_v &= a_{12}X_u + a_{22}X_v
\end{aligned}$$

where the a_{ij}, $i, j = 1, 2$, were obtained in Chapter 3 and the coefficients Γ^k_{ij}, $i, j = 1, 2$, are called the Christoffel symbols of S in the parametrization X. Since $X_{uv} = X_{vu}$, we conclude that $\Gamma^1_{12} = \Gamma^1_{21}$ and $\Gamma^2_{12} = \Gamma^2_{21}$; that is, the Christoffel symbols are symmetric relative to the lower indices.

To determine the Christoffel symbols, we take the inner product of the first four relations with X_u and X_v, obtaining the system

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{pmatrix} = \begin{pmatrix} \langle X_{uu}, X_u\rangle \\ \langle X_{uu}, X_v\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2}E_u \\ F_u - \frac{1}{2}E_v \end{pmatrix} \implies \begin{pmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{2}E_u \\ F_u - \frac{1}{2}E_v \end{pmatrix}$$

$$\begin{pmatrix} \Gamma^1_{12} \\ \Gamma^2_{12} \end{pmatrix} = \begin{pmatrix} \langle X_{uv}, X_u\rangle \\ \langle X_{uv}, X_v\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2}E_v \\ \frac{1}{2}G_u \end{pmatrix} \implies \begin{pmatrix} \Gamma^1_{21} \\ \Gamma^2_{21} \end{pmatrix} = \begin{pmatrix} \Gamma^1_{12} \\ \Gamma^2_{12} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{2}E_v \\ \frac{1}{2}G_u \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma^1_{22} \\ \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} \langle X_{vv}, X_u\rangle \\ \langle X_{vv}, X_v\rangle \end{pmatrix} = \begin{pmatrix} F_v - \frac{1}{2}G_u \\ \frac{1}{2}G_v \end{pmatrix} \implies \begin{pmatrix} \Gamma^1_{22} \\ \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} F_v - \frac{1}{2}G_u \\ \frac{1}{2}G_v \end{pmatrix}$$

where we have used

$$\langle X_{uu}, X_u\rangle = \frac{1}{2}\frac{\partial}{\partial u}\langle X_u, X_u\rangle = \frac{1}{2}E_u, \quad \langle X_{uu}, X_v\rangle = \frac{\partial}{\partial u}\langle X_u, X_v\rangle - \langle X_u, X_{vu}\rangle = F_u - \langle X_u, X_{uv}\rangle = F_u - \frac{1}{2}E_v$$

In particular, if X is an orthogonal parametrization, i.e. $F = \langle X_u, X_v\rangle = 0$ at each $p \in X(U)$, then

$$\begin{pmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{pmatrix} = \frac{1}{2(EG - F^2)}\begin{pmatrix} GE_u \\ -EE_v \end{pmatrix}$$
$$\begin{pmatrix} \Gamma^1_{12} \\ \Gamma^2_{12} \end{pmatrix} = \frac{1}{2(EG - F^2)}\begin{pmatrix} GE_v \\ EG_u \end{pmatrix}$$
$$\begin{pmatrix} \Gamma^1_{22} \\ \Gamma^2_{22} \end{pmatrix} = \frac{1}{2(EG - F^2)}\begin{pmatrix} GG_u \\ EG_v \end{pmatrix}$$

Thus, it is possible to solve the above system and to compute the Christoffel symbols in terms of the coefficients of the first fundamental form, E, F, G and their derivatives. Hence, all geometric concepts and properties expressed in terms of the Christoffel symbols are invariant under isometries.

Example Let S be a surface of revolution parametrized by

$$X(u,v) = (f(v)\cos u, f(v)\sin v, g(v)), \quad f(v) \neq 0.$$

Since

$$E = (f(v))^2, \quad F = 0, \quad G = (f'(v))^2 + (g'(v))^2,$$

we obtain

$$E_u = 0,\ E_v = 2ff',\ F_u = F_v = 0,\ G_u = 0,\ G_v = 2(f'f'' + g'g''),$$

and

$$\Gamma_{11}^1 = 0,\ \Gamma_{11}^2 = -\frac{ff'}{(f')^2 + (g')^2},\ \Gamma_{12}^1 = \frac{ff'}{f^2},\ \Gamma_{12}^2 = 0,\ \Gamma_{22}^1 = 0,\ \Gamma_{22}^2 = \frac{f'f'' + g'g''}{(f')^2 + (g')^2}.$$

Since $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is differentiable,

$$\begin{aligned}
& (X_{uu})_v = (X_{uv})_u, \\
\iff\ & \left(\Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + eN\right)_v = \left(\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + fN\right)_u \\
\iff\ & \Gamma_{11}^1 X_{uv} + \Gamma_{11}^2 X_{vv} + eN_v + (\Gamma_{11}^1)_v X_u + (\Gamma_{11}^2)_v X_v + e_v N \\
& = \Gamma_{12}^1 X_{uu} + \Gamma_{12}^2 X_{vu} + fN_u + (\Gamma_{12}^1)_u X_u + (\Gamma_{12}^2)_u X_v + f_u N \quad (*)
\end{aligned}$$

By equating the coefficients of X_v, and using

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{-1}{EG - F^2}\begin{pmatrix} eG - fF & -eF + fE \\ fG - gF & -fF + gE \end{pmatrix}.$$

we obtain the following formula for the Gaussian curvature K

$$\begin{aligned}
& \Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 + ea_{22} + (\Gamma_{11}^2)_v = \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 + fa_{21} + (\Gamma_{12}^2)_u \\
\iff\ & (\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^1\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 = ea_{22} - fa_{21} \\
\iff\ & (\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^1\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 = -E\frac{eg - f^2}{EG - F^2} \\
\iff\ & (\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^1\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 = -EK,
\end{aligned}$$

THEOREMA EGREGIUM (Gauss) The Gaussian curvature K of a surface is invariant by local isometries.

Remarks

- By equating the coefficients of X_u in equation $(*)$, we obtain another formula of the Gaussian curvature K.

$$(\Gamma_{12}^1)_u - (\Gamma_{11}^1)_v + \Gamma_{12}^1\Gamma_{11}^1 + \Gamma_{12}^2\Gamma_{12}^1 - \Gamma_{11}^1\Gamma_{12}^1 - \Gamma_{11}^2\Gamma_{22}^1 = (\Gamma_{12}^1)_u - (\Gamma_{11}^1)_v + \Gamma_{12}^2\Gamma_{12}^1 - \Gamma_{11}^2\Gamma_{22}^1 = FK.$$

- By equating the coefficients of N in equation $(*)$, we obtain

$$e_v - f_u = e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2 \quad (\dagger).$$

- By equating the coefficients of N in equation $(X_{vv})_u - (X_{vu})_v = 0$, we obtain

$$f_v - g_u = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12} \quad (\dagger\dagger).$$

Equations $(\dagger)$ and $(\dagger\dagger)$ are called Mainardi-Codazzi equations.

Theorem (Bonnet) Let E, F, G, e, f, g be differentiable functions, defined in an open set $V \subset \mathbb{R}^2$, with $E > 0$ and $G > 0$. Assume that

- the given functions satisfy formally the Gauss and Mainardi-Codazzi equations,
- and that $EG - F^2 > 0$.

Then,

- for every $q \in V$ there exists a neighborhood $U \subset V$ of q,
- and a diffeomorphism $X : U \to X(U) \subset \mathbb{R}^3$

such that the regular surface $X(U) \subset \mathbb{R}^3$ has E, F, G and e, f, g as coefficients of the first and second fundamental forms, respectively.

Furthermore, if U is connected and if

$$\bar{X} : U \to \bar{X}(U) \subset \mathbb{R}^3$$

is another diffeomorphism satisfying the same conditions, then there exist a translation T and a proper linear orthogonal transformation ρ in $\mathbb{R}^3$ such that

$$\bar{X} = T \circ \rho \circ X.$$

Remark In the following, we shall calculate the Christoffel symbols and Gaussian curvatures in terms of the metric tensor (g_{ij}) and its partial derivatives.

Let U be an open subset in the u_1u_2-plane, and $X : U \subset \mathbb{R}^2 \to S$ be a parametrization in the orientation of S. At each $p \in X(U)$, let $X_1 = X_{u_1}$, $X_2 = X_{u_2}$, and let

$$g_{11} = \langle X_1, X_1\rangle = E, \quad g_{12} = g_{21} = \langle X_1, X_2\rangle = F, \quad g_{22} = \langle X_2, X_2\rangle = G \iff \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

and

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = (g^{ij}) = (g_{ij})^{-1} = \frac{1}{\det(g_{ij})} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}.$$

Note that

$$\sum_{k=1}^{2} g^{mk} g_{k\ell} \overset{(\dagger)}{=} \delta_{m\ell} = \begin{cases} 1 & \text{if } m = \ell \\ 0 & \text{if } m \neq \ell \end{cases}.$$

Since X_1, X_2, $N \in \mathbb{R}^3$ are linearly independent, we may express vectors $X_{ij} = X_{u_iu_j} \in \mathbb{R}^3$ and $N_i = N_{u_i} \in T_pS$ as

$$X_{ij} \overset{(*)}{=} \sum_{k=1}^{2} \Gamma^k_{ij} X_k + h_{ij} N, \quad i, j = 1, 2, \quad \text{where } \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

$$N_i \overset{(**)}{=} \sum_{j=1}^{2} a_{ji} X_j, \quad i = 1, 2,$$

where

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \frac{1}{EG-F^2}\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} -g_{22} & g_{21} \\ g_{12} & -g_{11} \end{pmatrix},$$

as obtained in Chapter 3 and the coefficients Γ_{ij}^k, $i, j = 1, 2$, are called the Christoffel symbols of S in the parametrization X. Since $X_{ij} = X_{ji}$, we conclude that $\Gamma_{ij}^k = \Gamma_{ji}^k$; that is, the Christoffel symbols are symmetric relative to the lower indices.

To determine the Christoffel symbols, we take the inner product of the X_{ij} with X_k and use the definition of (g_{ij}) and (g^{ij}) to obtain

$$\langle X_{ij}, X_k\rangle = \frac{\partial}{\partial u_j}\langle X_i, X_k\rangle - \langle X_i, X_{kj}\rangle = \frac{\partial g_{ik}}{\partial u_j} - \langle X_i, X_{kj}\rangle = g_{ik,j} - \langle X_i, X_{kj}\rangle,$$

where $g_{ik,j} = \dfrac{\partial g_{ik}}{\partial u_j}$

$$\overset{(*)}{\iff} \langle \sum_{\ell=1}^{2} \Gamma_{ij}^{\ell} X_\ell, X_k\rangle = g_{ik,j} - \langle X_i, X_{kj}\rangle$$

$$\iff \sum_{\ell=1}^{2} \Gamma_{ij}^{\ell} g_{\ell k} = g_{ik,j} - \langle X_i, X_{kj}\rangle$$

$$\overset{(\dagger)}{\iff} \sum_{k=1}^{2}\sum_{\ell=1}^{2} \Gamma_{ij}^{\ell} g^{mk} g_{\ell k} = \sum_{k=1}^{2} g^{mk} g_{ik,j} - \sum_{k=1}^{2} g^{mk}\langle X_i, X_{kj}\rangle, \quad m = 1, 2$$

$$\overset{(\dagger)}{\iff} \sum_{\ell=1}^{2} \delta_{m\ell}\Gamma_{ij}^{\ell} = \sum_{k=1}^{2} g^{mk} g_{ik,j} - \sum_{k=1}^{2} g^{mk}\langle X_i, X_{kj}\rangle, \quad m = 1, 2$$

$$\overset{(\dagger)}{\iff} \Gamma_{ij}^{m} = \sum_{k=1}^{2} g^{mk} g_{ik,j} - \sum_{k=1}^{2} g^{mk}\langle X_i, X_{kj}\rangle, \quad m = 1, 2$$

$$\overset{\Gamma_{ij}^m = \Gamma_{ji}^m}{\iff} \Gamma_{ji}^{m} = \sum_{k=1}^{2} g^{mk} g_{jk,i} - \sum_{k=1}^{2} g^{mk}\langle X_j, X_{ki}\rangle, \quad m = 1, 2$$

$$\implies 2\Gamma_{ij}^{m} = \sum_{k=1}^{2} g^{mk}(g_{ik,j} + g_{jk,i}) - \sum_{k=1}^{2} g^{mk}\frac{\partial}{\partial u_k}\langle X_i, X_j\rangle = \sum_{k=1}^{2} g^{mk}(g_{ik,j} + g_{jk,i} - g_{ij,k}), \quad m = 1, 2$$

$$\implies \Gamma_{ij}^{m} = \frac{1}{2}\sum_{k=1}^{2} g^{mk}(g_{ik,j} + g_{jk,i} - g_{ij,k}), \quad m, i, j = 1, 2.$$

Since $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is differentiable,

$$(X_{ii})_j = (X_{ij})_i, \quad 1 \le i \ne j \le 2$$

$$\iff \left(\sum_{k=1}^{2} \Gamma_{ii}^k X_k + h_{ii}N\right)_j = \left(\sum_{k=1}^{2} \Gamma_{ij}^k X_k + h_{ij}N\right)_i$$

$$\iff \sum_{k=1}^{2} (\Gamma_{ii}^k)_j X_k + \sum_{k=1}^{2} \Gamma_{ii}^k X_{kj} + h_{ii,j}N + h_{ii}N_j = \sum_{k=1}^{2} (\Gamma_{ij}^k)_i X_k + \sum_{k=1}^{2} \Gamma_{ij}^k X_{ki} + h_{ij,i}N + h_{ij}N_i,$$

where $(\Gamma_{ii}^k)_j = \dfrac{\partial \Gamma_{ii}^k}{\partial u_j}$, $h_{ij,i} = \dfrac{\partial h_{ij}}{\partial u_i}$

$$\begin{aligned}
\iff\quad & \sum_{k=1}^{2}(\Gamma_{ii}^{k})_j X_k + \sum_{k,\,\ell=1}^{2}\Gamma_{ii}^{k}\Gamma_{kj}^{\ell}X_\ell + \sum_{k=1}^{2}\Gamma_{ii}^{k}h_{kj}N + h_{ii,j}N + \sum_{k=1}^{2}h_{ii}a_{kj}X_k \\
& = \sum_{k=1}^{2}(\Gamma_{ij}^{k})_i X_k + \sum_{k,\,\ell=1}^{2}\Gamma_{ij}^{k}\Gamma_{ki}^{\ell}X_\ell + \sum_{k=1}^{2}\Gamma_{ij}^{k}h_{ki}N + h_{ij,i}N + \sum_{k=1}^{2}h_{ij}a_{ki}X_k \\
\iff\quad & \sum_{k=1}^{2}(\Gamma_{ii}^{k})_j X_k + \sum_{\ell,\,k=1}^{2}\Gamma_{ii}^{\ell}\Gamma_{\ell j}^{k}X_k + \sum_{k=1}^{2}\Gamma_{ii}^{k}h_{kj}N + h_{ii,j}N + \sum_{k=1}^{2}h_{ii}a_{kj}X_k \quad k\leftrightarrow\ell \text{ in double sum} \\
& = \sum_{k=1}^{2}(\Gamma_{ij}^{k})_i X_k + \sum_{\ell,\,k=1}^{2}\Gamma_{ij}^{\ell}\Gamma_{\ell i}^{k}X_k + \sum_{k=1}^{2}\Gamma_{ij}^{k}h_{ki}N + h_{ij,i}N + \sum_{k=1}^{2}h_{ij}a_{ki}X_k \quad k\leftrightarrow\ell \text{ in double sum} \\
\Longrightarrow\quad & 0 = \sum_{k=1}^{2}\left[(\Gamma_{ij}^{k})_i - (\Gamma_{ii}^{k})_j + \sum_{\ell=1}^{2}\Gamma_{ij}^{\ell}\Gamma_{\ell i}^{k} - \sum_{\ell=1}^{2}\Gamma_{ii}^{\ell}\Gamma_{\ell j}^{k} + h_{ij}a_{ki} - h_{ii}a_{kj}\right]X_k \\
& \quad + \left(h_{ij,i} - h_{ii,j} + \sum_{k=1}^{2}\Gamma_{ij}^{k}h_{ki} - \sum_{k=1}^{2}\Gamma_{ii}^{k}h_{kj}\right)N \\
\iff\quad & (\Gamma_{ij}^{k})_i - (\Gamma_{ii}^{k})_j + \sum_{\ell=1}^{2}\Gamma_{ij}^{\ell}\Gamma_{\ell i}^{k} - \sum_{\ell=1}^{2}\Gamma_{ii}^{\ell}\Gamma_{\ell j}^{k} = h_{ii}a_{kj} - h_{ij}a_{ki} \quad 1\le i\ne j\le 2, \\
\text{and}\quad & h_{ij,i} - h_{ii,j} + \sum_{k=1}^{2}\Gamma_{ij}^{k}h_{ki} - \sum_{k=1}^{2}\Gamma_{ii}^{k}h_{kj} = 0 \quad 1\le i\ne j\le 2 \quad \text{called Mainardi-Codazzi equations}
\end{aligned}$$

Since $h_{ij} = h_{ji}$, $1 \le i \ne j \le 2$,

$$\begin{pmatrix} h^{11} & h^{12} \\ h^{21} & h^{22}\end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22}\end{pmatrix}^{-1} = \frac{1}{eg-f^2}\begin{pmatrix} h_{22} & -h_{21} \\ -h_{12} & h_{11}\end{pmatrix} = \frac{1}{eg-f^2}\begin{pmatrix} h_{22} & -h_{12} \\ -h_{21} & h_{11}\end{pmatrix},$$

we have

$$\begin{aligned}
h_{ii}a_{kj} - h_{ij}a_{ki} &= (eg-f^2)[h^{jj}a_{kj} + h^{ji}a_{ki}] = (eg-f^2)\sum_{\ell=1}^{2}h^{j\ell}a_{k\ell} \\
&= (eg-f^2)\times\left[\begin{pmatrix} h^{11} & h^{12} \\ h^{21} & h^{22}\end{pmatrix}\cdot\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22}\end{pmatrix}\right]_{jk} \quad (\text{the } jk\text{-entry of } (h^{mn})\cdot(a_{pq})) \\
&= \frac{eg-f^2}{EG-F^2}\times\left[\begin{pmatrix} h^{11} & h^{12} \\ h^{21} & h^{22}\end{pmatrix}\cdot\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22}\end{pmatrix}\cdot\begin{pmatrix} -g_{22} & g_{21} \\ g_{12} & -g_{11}\end{pmatrix}\right]_{jk} \\
&= K\times\begin{pmatrix} -g_{22} & g_{21} \\ g_{12} & -g_{11}\end{pmatrix}_{jk} = K\times\begin{pmatrix} -G & F \\ F & -E\end{pmatrix}_{jk}
\end{aligned}$$

and the Gauss curvature formulas

$$(\Gamma_{ij}^{k})_i - (\Gamma_{ii}^{k})_j + \sum_{\ell=1}^{2}\Gamma_{ij}^{\ell}\Gamma_{\ell i}^{k} - \sum_{\ell=1}^{2}\Gamma_{ii}^{\ell}\Gamma_{\ell j}^{k} = h_{ii}a_{kj} - h_{ij}a_{ki} = K\times\begin{pmatrix} -g_{22} & g_{21} \\ g_{12} & -g_{11}\end{pmatrix}_{jk},$$

where $\Gamma_{ij}^{k} = \dfrac{1}{2}\sum_{\ell=1}^{2}g^{k\ell}(g_{i\ell,j} + g_{j\ell,i} - g_{ij,\ell})$, $i,\ j,k = 1,\ 2$.

In particular, we obtain the following when

$$j=k=2,\ i=1 \implies (\Gamma_{12}^2)_1-(\Gamma_{11}^2)_2+\sum_{\ell=1}^2\Gamma_{12}^\ell\Gamma_{\ell 1}^2-\sum_{\ell=1}^2\Gamma_{11}^\ell\Gamma_{\ell 2}^2=-h_{12}a_{21}+h_{11}a_{22}=-EK,$$

$$j=k=1,\ i=2 \implies (\Gamma_{21}^1)_2-(\Gamma_{22}^1)_1+\sum_{\ell=1}^2\Gamma_{21}^\ell\Gamma_{\ell 2}^1-\sum_{\ell=1}^2\Gamma_{22}^\ell\Gamma_{\ell 1}^1=h_{22}a_{k1}-h_{21}a_{k2}=-GK,$$

$$j\neq k=1,\ i=1 \implies (\Gamma_{12}^1)_1-(\Gamma_{11}^1)_2+\sum_{\ell=1}^2\Gamma_{12}^\ell\Gamma_{\ell 1}^1-\sum_{\ell=1}^2\Gamma_{11}^\ell\Gamma_{\ell 2}^1=-h_{12}a_{11}+h_{11}a_{12}=FK.$$

Example Let $X(u_1,u_2)$ be an orthogonal parametrization (that is, $F=g_{12}=g_{21}=0$) of a neighborhood of an oriented surface S. Let $g^{ik}_{,m}=\dfrac{\partial g^{ik}}{\partial u_m}$ and $g_{k\ell,m}=\dfrac{\partial g_{k\ell}}{\partial u_m}$. Since

$$\sum_{k=1}^2 g^{ik}g_{kj}=\delta_{ij}=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases},\quad g_{12}=g_{21}=0\quad\text{and}\quad g^{12}=g^{21}=0,$$

and using $\Gamma_{ij}^k=\dfrac{1}{2}\displaystyle\sum_{k=1}^2 g^{k\ell}(g_{j\ell,i}+g_{\ell i,j}-g_{ij,\ell})=\dfrac{1}{2}g^{kk}(g_{jk,i}+g_{ki,j}-g_{ij,k})$, we have

$$\sum_{k=1}^2 g^{ik}_{,m}g_{k\ell}+\sum_{k=1}^2 g^{ik}g_{k\ell,m}=0 \implies \sum_{\ell=1}^2 g^{\ell j}\sum_{k=1}^2 g^{ik}_{,m}g_{k\ell}+\sum_{\ell=1}^2 g^{\ell j}\sum_{k=1}^2 g^{ik}g_{k\ell,m}=0$$

$$\implies g^{ij}_{,m}=\sum_{k=1}^2 g^{ik,m}\delta_{jk}=-\sum_{k,\,\ell=1}^2 g^{ik}g_{k\ell,m}g^{\ell j}\implies g^{ij}_{,m}=-g^{ii}\,g_{ij,m}\,g^{jj}\implies g^{ii}_{,m}=-g^{ii}\,g_{ii,m}\,g^{ii},$$

$$\implies \Gamma_{12}^2=\frac{1}{2}g^{22}\,g_{22,1},\quad \Gamma_{11}^2=-\frac{1}{2}g^{22}\,g_{11,2},\quad \Gamma_{12}^1=\frac{1}{2}g^{11}\,g_{11,2},\quad \Gamma_{11}^1=\frac{1}{2}g^{11}\,g_{11,1},\quad \Gamma_{22}^2=\frac{1}{2}g^{22}\,g_{22,2}$$

and

$$\left(\Gamma_{12}^2\right)_1-\left(\Gamma_{11}^2\right)_2+\sum_{\ell=1}^2\Gamma_{12}^\ell\Gamma_{\ell 1}^2-\sum_{\ell=1}^2\Gamma_{11}^\ell\Gamma_{\ell 2}^2=-Kg_{11}$$

$$\iff \frac{1}{2}\left(g^{22}\,g_{22,1}\right)_1+\frac{1}{2}\left(g^{22}\,g_{11,2}\right)_2-\frac{1}{4}\left(g^{11}\,g_{11,2}\,g^{22}\,g_{11,2}\right)+\frac{1}{4}\left(g^{22}\,g_{22,1}\,g^{22}\,g_{22,1}\right)$$
$$-\frac{1}{4}\left(g^{11}\,g_{11,1}\,g^{22}\,g_{22,1}\right)+\frac{1}{4}\left(g^{22}\,g_{11,2}\,g^{22}\,g_{22,2}\right)=-Kg_{11}$$

$$\iff \left(\frac{g_{22,1}}{2\sqrt{g_{11}g_{22}}}\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}}\right)_1+\left(\frac{g_{11,2}}{2\sqrt{g_{11}g_{22}}}\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}}\right)_2+\frac{(g_{22,1})^2+g_{11,2}g_{22,2}}{4\,(g_{22})^2}-\frac{(g_{11,2})^2+g_{11,1}g_{22,1}}{4\,g_{11}\,g_{22}}=-Kg_{11}$$

$$\iff \left(\frac{g_{22,1}}{2\sqrt{g_{11}g_{22}}}\right)_1\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}}+\left(\frac{g_{11,2}}{2\sqrt{g_{11}g_{22}}}\right)_2\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}}=-Kg_{11}$$

$$\iff K=-\frac{1}{2\sqrt{g_{11}\,g_{22}}}\left[\left(\frac{g_{22,1}}{\sqrt{g_{11}g_{22}}}\right)_1+\left(\frac{g_{11,2}}{\sqrt{g_{11}g_{22}}}\right)_2\right]$$

Parallel Transport. Geodesics.

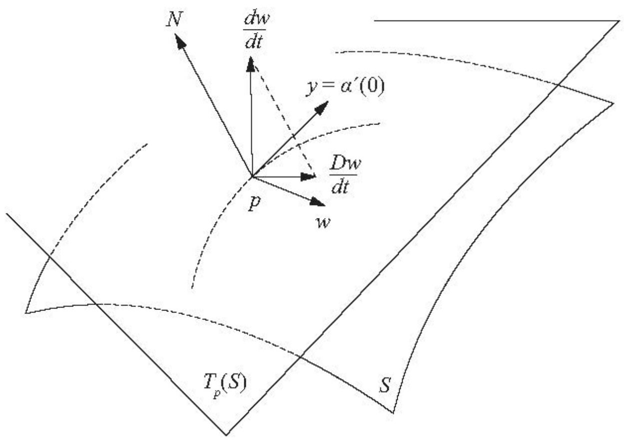
Definition Let $w:U\to\mathbb{R}^3$ be a differentiable tangent vector field in an open set $U\subset S$ and $p\in U$. Let $y\in T_pS$. Consider a parametrized curve

$$\alpha:(-\varepsilon,\varepsilon)\to U,\quad \text{with } \alpha(0)=p \text{ and } \alpha'(0)=y,$$

and let $w(t) = w(\alpha(t)) \in T_{\alpha(t)}S$, $t \in (-\varepsilon, \varepsilon)$, be the restriction of the vector field w to the curve α.

Then the covariant derivative at p of the vector field w relative to the vector y, denoted $\frac{Dw}{dt}(0)$ or $D_y w(p)$, is defined to be the normal projection of $\frac{dw}{dt}(0)$ onto the plane T_pS, i.e.

$$\frac{Dw}{dt}(0) = \frac{dw}{dt}(0) - \langle \frac{dw}{dt}(0), N(p) \rangle N.$$



In terms of a parametrization $X(u_1, u_2)$ of $U \subset S$ at p, let $X(u_1(t), u_2(t)) = \alpha(t) \subset S$ and

$$w(t) = a_1(u_1(t), u_2(t))X_{u_1} + a_2(u_1(t), u_2(t))X_{u_2} = a_1(t)X_1 + a_2(t)X_2 = \sum_{i=1}^{2} a_i X_i \ \in T_{\alpha(t)}S$$

be the expression of $\alpha(t)$ and $w(t)$ in the parametrization $X(u, v)$, respectively. Then

$$\frac{dw}{dt} = \sum_{i,j=1}^{2} a_i X_{ij} u'_j + \sum_{i=1}^{2} a'_i X_i = \sum_{i,j,k=1}^{2} a_i u'_j \Gamma^k_{ij} X_k + \sum_{i,j=1}^{2} a_i u'_j h_{ij} N + \sum_{k=1}^{2} a'_k X_k$$

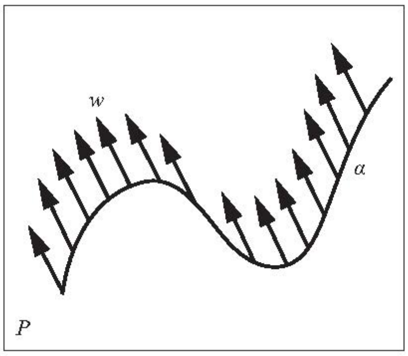
and the covariant derivative of w at t is given by

$$\begin{aligned} \frac{Dw}{dt} &= \sum_{i,j,k=1}^{2} a_i u'_j \Gamma^k_{ij} X_k + \sum_{i=1}^{2} a'_i X_i \\ &= \sum_{k=1}^{2} \left(a'_k + \sum_{i,j=1}^{2} \Gamma^k_{ij} a_i u'_j \right) X_k \ \in T_{\alpha(t)}S \end{aligned}$$

Note that the covariant differentiation $\frac{Dw}{dt}$ depends only on the vector (u'_1, u'_2), the coordinates of $\alpha'(t)$ in the basis $\{X_1, X_2\}$, and not on the curve α. Also since it depends only on the Christoffel symbols, that is, the first fundamental form of the surface, the covariant differentiation $\frac{Dw}{dt}$ is a concept of intrinsic geometry.

Definition A vector field $w \in T_\alpha S$ along a parametrized curve $\alpha : I \to S$ is said to be parallel if $\dfrac{Dw}{dt} = 0$ for every $t \in I$.

Example In a plane P, since $\Gamma_{ij}^k = 0$, $1 \le i, j, k \le 2$, the notion of parallel field $w = a_1 X_1 + a_2 X_2$ along a parametrized curve $\alpha \subset P$ reduces to that of a constant field, i.e. $a_1' = a_2' = 0$, along α; that is, the length of the vector and its angle with a fixed direction are constant.



Those properties are partially reobtained on any surface as the following proposition shows.

Proposition Let $w, v \in T_\alpha S$ be parallel vector fields along $\alpha : I \to S$. Then $\langle w(t), v(t) \rangle$ is constant for all $t \in I$. In particular, the lengths $|w(t)|$ and $|v(t)|$ are constant, and the angle $\angle(v(t), w(t))$ between $w(t)$, $v(t) \in T_{\alpha(t)} S$ is constant for all $t \in I$.

Proof Since $w(t)$, $v(t) \in T_{\alpha(t)} S$ and $\dfrac{Dw}{dt} = \dfrac{Dv}{dt} = 0$, we have

$$\frac{d}{dt}\langle w(t), v(t)\rangle = \langle \frac{dw}{dt}, v(t)\rangle + \langle w(t), \frac{dv}{dt}\rangle = \langle \frac{Dw}{dt}, v(t)\rangle + \langle w(t), \frac{Dv}{dt}\rangle = 0,$$

and $\langle w(t), v(t) \rangle =$ constant for all $t \in I$ and for any parallel vector fields w and v along α.

Proposition Let $\alpha : I \to S$ be a parametrized curve in S and let $w_0 \in T_{\alpha(t_0)} S$, $t_0 \in I$. Then there exists a unique parallel vector field $w(t) = a_1(t) X_1(u_1(t), u_2(t)) + a_2(t) X_2(u_1(t), u_2(t))$ along $\alpha(t)$, with $w(t_0) = w_0$, i.e. there is a unique solution to the initial-value problem

$$a_k' + \sum_{i,j=1}^{2} \Gamma_{ij}^k a_i u_j' = 0, \quad k = 1, 2, \text{ with } a_1 X_1 + a_2 X_2|_{t=t_0} = w(t_0) = w_0.$$

Definition Let $\alpha : I \to S$ be a parametrized curve and $w_0 \in T_{\alpha(t_0)} S$, $t_0 \in I$. Let w be a paralle vector field along α, with $w(t_0) = w_0$. The vector $w(t_1)$, $t_1 \in I$, is called the parallel transport of w_0 along α at the point t_1.

Definition A nonconstant, parametrized curve $\gamma : I \to S$ is said to be geodesic at $t \in I$ if the field of its tangent vectors $\gamma'(t)$ is parallel along γ at t; that is

$$\frac{D\gamma'(t)}{dt} = 0\,;$$

γ is a parametrized geodesic if it is geodesic for all $t \in I$, i.e. $\gamma(t) = X(u_1(t), u_2(t))$, $t \in I$ is a geodesic if $\gamma'(t) = u_1' X_1 + u_2' X_2$ satisfies the geodesic equations

$$\frac{D\gamma'(t)}{dt} = 0 \iff u_k'' + \sum_{i,j=1}^{2} \Gamma_{ij}^k u_i' u_j', \quad k = 1,\, 2. \quad (*)$$

Examples

(1) If S is a plane, then S can be parametrized by $X(u_1,u_2)$ with $X_{ij} = X_{u_iu_j} = 0 \in \mathbb{R}^3$ everywhere in S, $1 \le i,j \le 2$. This implies that $X_1 = X_{u_1}$ and $X_2 = X_{u_2}$ are constant vector in S, $\Gamma^k_{ij} = 0$ for all $1 \le i,j,k \le 2$, and $\gamma(t) = X(u_1(t),u_2(t))$ is a geodesic in a plane S if

$$u_k''(t) = 0,\ \forall\, t \in I \implies u_k'(t) = c_k \text{ (a constant) } \forall\, t \in I \implies u_k(t) = c_k t + d_k\ \forall\, t \in I,\ \ k = 1,\,2$$

Hence γ is a geodesic in a plane S if and only if γ is a straight line in S.

(2) Let $\gamma(u_2) = (f(u_2), 0, g(u_2))$, $f(u_2) \neq 0$, $a < u_2 < b$, be a regular curve and S be a surface of revolution with the parametrization

$$X(u_1,u_2) = (f(u_2)\cos u_1, f(u_2)\sin u_1, g(u_2)), \quad 0 < u_1 < 2\pi,\ a < u_2 < b.$$

Then the matrix (g_{ij}) and its inverse (g^{ij}) of the first fundamental form $\sum_{i,j=1}^{2} g_{ij}u_i'u_j'$ are given by

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} f^2 & 0 \\ 0 & (f')^2 + (g')^2 \end{pmatrix} \iff \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} f^{-2} & 0 \\ 0 & \left[(f')^2 + (g')^2\right]^{-1} \end{pmatrix}$$

where f and g are functions of u_2 and the Christoffel symbols $\Gamma^k_{ij} = \frac{1}{2}\sum_{\ell=1}^{2} g^{k\ell}(g_{j\ell,i} + g_{\ell i,j} - g_{ij,\ell})$ are given by

$$\Gamma^1_{ij} = \frac{1}{2} f^{-2}(g_{j1,i} + g_{1i,j} - g_{ij,1}) \quad \text{and} \quad \Gamma^2_{ij} = \frac{1}{2}\left[(f')^2 + (g')^2\right]^{-1}(g_{j2,i} + g_{2i,j} - g_{ij,2})$$

and

$$\begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} \\ \Gamma^1_{21} & \Gamma^1_{22} \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{ff'}{f^2} \\ \dfrac{ff'}{f^2} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Gamma^2_{11} & \Gamma^2_{12} \\ \Gamma^2_{21} & \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} -\dfrac{ff'}{(f')^2 + (g')^2} & 0 \\ 0 & \dfrac{f'f'' + g'g''}{(f')^2 + (g')^2} \end{pmatrix}$$

this implies that $X(u_1(t),u_2(t))$ is a geodesic of the surface of revolution S if u_1, u_2 satisfy the system of equations

$$u_1'' + \frac{2ff'}{f^2}u_1'u_2' = 0 \quad \text{and} \quad u_2'' - \frac{ff'}{(f')^2 + (g')^2}(u_1')^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}(u_2')^2 = 0, \qquad (\dagger\dagger)$$

where $u_k' = \dfrac{du_k}{dt}$, $f' = \dfrac{df}{du_2}$ and $g' = \dfrac{dg}{du_2}$.

If the meridian $\gamma(s) = \{X(u_1,u_2) \mid u_1 = \text{constant},\ u_2 = u_2(s)\}$ is parametrized by arc length s, then the 1st equation of $(\dagger\dagger)$ holds, and, since $\gamma'(s) = X_1u_1' + X_2u_2' = X_2u_2'$,

$$1 = \langle \gamma'(s), \gamma'(s)\rangle = I_p(\gamma'(s)) = \langle X_1u_1' + X_2u_2', X_1u_1' + X_2u_2'\rangle = g_{22}\,(u_2')^2 = \left[(f')^2 + (g')^2\right](u_2')^2,$$

we have

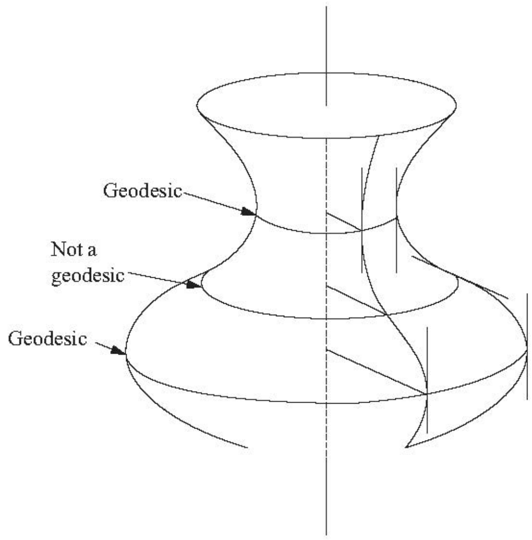
$$(u_2')^2 \overset{(*)}{=} \frac{1}{(f')^2 + (g')^2} \implies u_2'\,u_2'' = -\frac{f'f'' + g'g''}{\left[(f')^2 + (g')^2\right]^2}\,u_2'$$

by differentiating both sides with respect to s and using the Chain Rule to get $\dfrac{d}{ds}f' = f''\,u_2'$ and $\dfrac{d}{ds}g' = g''\,u_2'$. Multiplying both sides by u_2', we get

$$(u_2')^2\,u_2'' = -\frac{f'f''+g'g''}{\big[(f')^2+(g')^2\big]^2}\,(u_2')^2 \overset{(*)}{\Longrightarrow} u_2'' = -\frac{f'f''+g'g''}{(f')^2+(g')^2}(u_2')^2 \quad \text{the } 2^{\text{nd}} \text{ equation of } (\dagger\dagger)$$

and this implies that arc length parametrized meridians are geodesics.
If the parallel $\gamma(s) = \{X(u_1,u_2) \mid u_2 = \text{constant},\ u_1 = u_1(s)\}$ is parametrized by arc length s, since $1 = I_p(\gamma'(s)) = (f(u_2))^2(u_1')^2$, we have $(u_1')^2 = 1/f(u_2) = $ constant $\neq 0$ which implies that $2u_1'u_1'' = 0 \implies u_1'' = 0$, i.e. the 1^{st} equation of $(\dagger\dagger)$ holds, so the arc length parametrized parallels are geodesics if it satisfies the 2^{nd} equation of $(\dagger\dagger)$

$$\frac{ff'}{(f')^2+(g')^2}(u_1')^2 = 0 \implies f' = 0 \quad \text{since } f \neq 0,\ u_1' \neq 0.$$



Definition Let w be a differentiable field of unit vectors along a parametrized curve $\alpha : I \to S$ on an oriented surface S. Since $w(t)$, $t \in I$, is a unit vector field,

$$\frac{dw}{dt}(t) \perp w(t) \implies \frac{Dw}{dt} = \left[\frac{Dw}{dt}\right](N \wedge w(t)),$$

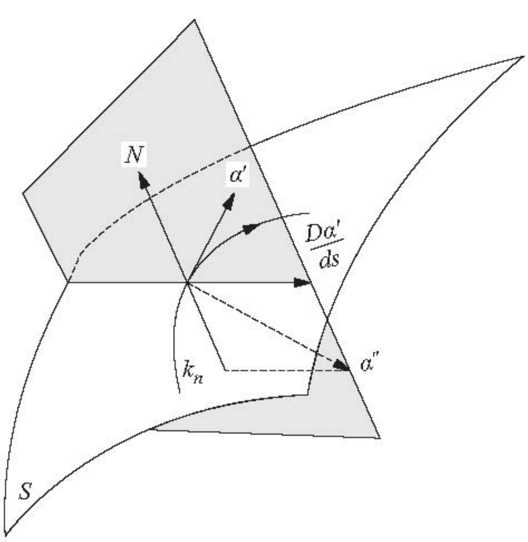
where the real number $\left[\dfrac{Dw}{dt}\right]$ is called the algebraic value of the covariant derivative of w at t.

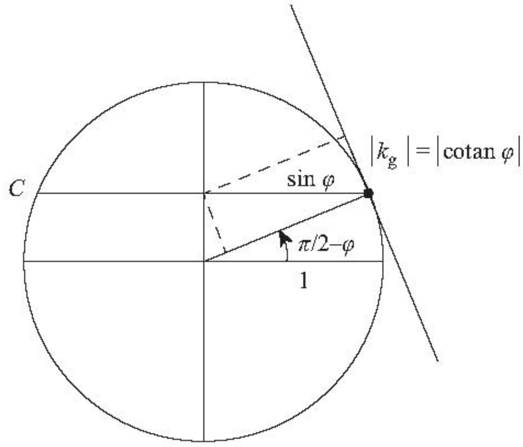
Definition Let C be an oriented regular curve contained in an oriented surface S, and let $\alpha(s)$ be a parametrization of C, in a neighborhood of $p \in S$, by the arc length s. The algebraic value of the covariant derivative of $\alpha'(s)$ at p, $\left[\dfrac{D\alpha'(s)}{ds}\right] = k_g$ is called the geodesic curvature of C at p.

Remark The geodesics which are regular curves are thus characterized as curves whose geodesic curvature is zero and note that the geodesic curvature of $C \subset S$ changes sign when we change the orientation of either C or S.

Furthermore, since $\dfrac{d\alpha'(s)}{ds} = k\,n(s)$, where $n(s)$ is the unit normal vector to C at $\alpha(s)$, and since

$$\frac{D\alpha'(s)}{ds} = \frac{d\alpha'(s)}{ds} - \langle \frac{d\alpha'(s)}{ds}, N\rangle N = k\,n(s) - k\langle n(s), N\rangle N = k\,n(s) - k_n N \implies k_g^2 + k_n^2 = k^2.$$





Geodesic curvature of a parallel on a unit sphere.

Example The absolute value of the geodesic curvature k_g of a parallel C of colatitude φ in a unit sphere S^2 can be computed from the relation

$$\frac{1}{\sin^2\varphi} = k_n^2 + k_g^2 = 1 + k_g^2 \implies k_g^2 = \frac{1}{\sin^2\varphi} - 1 = \frac{\cos^2\varphi}{\sin^2\varphi} = \cot^2\varphi,$$

where the sign of k_g depends on the orientations of S^2 and C.

Lemma Let a and b be differentiable in I with $a^2 + b^2 = 1$ and φ_0 be such that $a(t_0) = \cos\varphi_0$, $b(t_0) = \sin\varphi_0$. Then the differentiable function φ defined by

$$\varphi(t) = \varphi_0 + \int_{t_0}^{t} (ab' - ba')\,du$$

satisfies

$$\cos\varphi(t) = a(t), \quad \sin\varphi(t) = b(t), \quad \text{for } t \in I, \quad \text{and} \quad \varphi(t_0) = \varphi_0.$$

Proof It suffices to show that the function

$$\begin{aligned} & (a-\cos\varphi)^2 + (b-\sin\varphi)^2 = 0 \quad \forall t \in I,\\ \overset{a^2+b^2=1}{\iff}\ & 2 - 2(a\cos\varphi + b\sin\varphi) = 0 \quad \forall t \in I,\\ \iff\ & A = a\cos\varphi + b\sin\varphi = 1 \quad \forall t \in I. \end{aligned}$$

Since $a^2 + b^2 = 1$ for all $t \in I$, $aa' = -bb'$ and, by the definition of φ, we have

$$\begin{aligned} A' &= -a\,(\sin\varphi)\,\varphi' + b\,(\cos\varphi)\,\varphi' + a'\,\cos\varphi + b'\,\sin\varphi\\ &= -a\,(\sin\varphi)\,(ab' - ba') + b\,(\cos\varphi)\,(ab' - ba') + a'\,\cos\varphi + b'\,\sin\varphi\\ &\overset{aa'=-bb'}{=} -b'\,(\sin\varphi)\,(a^2+b^2) - a'\,(\cos\varphi)\,(a^2+b^2) + a'\,\cos\varphi + b'\,\sin\varphi\\ &\overset{a^2+b^2=1}{=} 0 \quad \forall t \in I. \end{aligned}$$

Therefore, $A(t) = \text{const.}$, and since $A(t_0) = 1$, the lemma is proved.

Remark Note that the differentiable function ψ defined by $\psi(t) = \varphi_0 - \int_{t_0}^{t} (ab' - ba')\, du$ satisfies

$$\cos\psi(t) = a(t), \quad -\sin\psi(t) = b(t), \quad \text{for } t \in I, \quad \psi(t_0) = \varphi_0, \quad \text{and} \quad \psi'(t) = -\varphi'(t).$$

Lemma Let v and w be two differentiable vector fields along the curve $\alpha : I \to S$, with $|w(t)| = |v(t)| = 1$, $t \in I$. Then

$$\left[\frac{Dw}{dt}\right] - \left[\frac{Dv}{dt}\right] = \frac{d\varphi}{dt},$$

where φ is one of the differentiable determinations of the angle from v to w, as given by the preceding Lemma.

Proof Since $|w(t)| = |v(t)| = 1$, for all $t \in I$,

$$\begin{aligned}
& \langle v(t), w(t)\rangle = \cos\varphi(t) \quad \forall\, t \in I \\
\Longrightarrow\quad & \langle \frac{dv}{dt}, w\rangle + \langle v, \frac{dw}{dt}\rangle = -\sin\varphi\,\varphi' \\
\Longrightarrow\quad & \langle \frac{Dv}{dt}, w\rangle + \langle v, \frac{Dw}{dt}\rangle = -\sin\varphi\,\varphi' \\
\Longleftrightarrow\quad & \left[\frac{Dv}{dt}\right]\langle N\wedge v, w\rangle + \left[\frac{Dw}{dt}\right]\langle v, N\wedge w\rangle = \left[\frac{Dv}{dt}\right]\langle N\wedge v, w\rangle - \left[\frac{Dw}{dt}\right]\langle w, N\wedge v\rangle = -\sin\varphi\,\varphi' \\
\Longleftrightarrow\quad & \left(\left[\frac{Dv}{dt}\right] - \left[\frac{Dw}{dt}\right]\right)\langle N\wedge v, w\rangle = \left(\left[\frac{Dv}{dt}\right] - \left[\frac{Dw}{dt}\right]\right)\langle N, v\wedge w\rangle = -\sin\varphi\,\varphi' \\
\Longleftrightarrow\quad & \left(\left[\frac{Dv}{dt}\right] - \left[\frac{Dw}{dt}\right]\right)(-\sin\varphi) = (-\sin\varphi)\,\varphi' \quad \text{if necessary switch orientation of angle } \angle(v, w)
\end{aligned}$$

If $\varphi \neq 0$, then $\sin\varphi \neq 0$ and we obtain that $\left[\frac{Dw}{dt}\right] - \left[\frac{Dv}{dt}\right] = \frac{d\varphi}{dt}$.

If $\varphi = 0$ at p, either $\varphi \equiv 0$ in a neighborhood U of p, or there exists a sequence $(p_n) \to p$ with $\varphi(p_n) \neq 0$. In the first case, $\varphi' \equiv 0$ in U, $v = w$ and the Lemma holds trivially. In the second case, since $\left[\frac{Dw}{dt}\right] - \left[\frac{Dv}{dt}\right] = \frac{d\varphi}{dt}$ at p_n, the Lemma holds by continuity.

Remarks

(a) In particular, if

- C is a regular oriented curve on S,
- $\alpha(s)$ is a parametrization by the arc length s of C at $p \in C$,
- $v(s)$ is a parallel field along $\alpha(s)$,
- $w(s) = \alpha'(s)$,

then

$$k_g(s) = \left[\frac{D\alpha'(s)}{ds}\right] = \left[\frac{D\alpha'(s)}{ds}\right] - \left[\frac{Dv(s)}{ds}\right] = \frac{d\varphi}{ds}, \quad \text{where } \varphi(s) = \angle(v(s), \alpha'(s)).$$

In other words, the geodesic curvature k_g is the rate of change of the angle that the tangent to the curve makes with a parallel direction along the curve. In the case of the plane, the parallel direction is fixed and the geodesic curvature k_g reduces to the usual curvature k.

(b) **Proposition** If $X(u_1,u_2)$ is an orthogonal parametrization (that is, $F = g_{12} = g_{21} = 0$) of a neighborhood of an oriented surface S, and $w(t)$ is a differentiable field of unit vectors along the curve $X(u_1(t),u_2(t))$, then

$$\left[\frac{Dw}{dt}\right] = \frac{1}{2\sqrt{g}}\,(g_{22,1}\,u_2' - g_{11,2}\,u_1') + \frac{d\varphi}{dt} = \frac{1}{2\sqrt{EG}}\,(G_{u_1}\,u_2' - E_{u_2}\,u_1') + \frac{d\varphi}{dt}, \qquad (\dagger)$$

where $\varphi(t)$ is the angle from X_1 to $w(t)$ in the given orientation.

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle X_1,X_1\rangle & \langle X_1,X_2\rangle \\ \langle X_2,X_1\rangle & \langle X_2,X_2\rangle \end{pmatrix}, \quad \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1}, \quad g = \det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

Proof Since $X(u_1,u_2)$ is an orthogonal parametrization, we have

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}, \quad \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} g^{11} & 0 \\ 0 & g^{22} \end{pmatrix} = \begin{pmatrix} g_{11}^{-1} & 0 \\ 0 & g_{22}^{-1} \end{pmatrix}$$

and since $\Gamma^k_{ij} = \dfrac{1}{2}\displaystyle\sum_{\ell=1}^{2} g^{k\ell}(g_{j\ell,i} + g_{\ell i,j} - g_{ij,\ell}) = \dfrac{1}{2}g^{kk}(g_{jk,i} + g_{ki,j} - g_{ij,k})$, we have

$$\begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} \\ \Gamma^1_{21} & \Gamma^1_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}g^{11}\,g_{11,1} & \frac{1}{2}g^{11}\,g_{11,2} \\ \frac{1}{2}g^{11}\,g_{11,2} & -\frac{1}{2}g^{11}\,g_{22,1} \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} \Gamma^2_{11} & \Gamma^2_{12} \\ \Gamma^2_{21} & \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}g^{22}\,g_{11,2} & \frac{1}{2}g^{22}\,g_{22,1} \\ \frac{1}{2}g^{22}\,g_{22,1} & \frac{1}{2}g^{22}\,g_{22,2} \end{pmatrix}$$

Let $e_i(t) = e_i(u_1(t),u_2(t)) = \dfrac{X_i}{\sqrt{g_{ii}}}$, $i = 1,\,2$, be the field of unit vectors $e_i = \dfrac{X_i}{\sqrt{g_{ii}}}$ restricted to the curve $X(u_1(t),u_2(t))$ with $e_1\wedge e_2 = N$, the given orientation of S, and set $v(t) = e_1(t)$ in the preceding Lemma, we get

$$\left[\frac{Dw}{dt}\right] = \left[\frac{De_1}{dt}\right] + \frac{d\varphi}{dt} = \left[\frac{D(X_1/\sqrt{g_{11}})}{dt}\right] + \frac{d\varphi}{dt} = \left[\frac{D(X_{u_1}/\sqrt{E})}{dt}\right] + \frac{d\varphi}{dt}.$$

Now

$$\begin{aligned}
\frac{D(X_1/\sqrt{g_{11}})}{dt} &= \sum_{j=1}^{2}\frac{\partial}{\partial u_j}\left(\frac{1}{\sqrt{g_{11}}}\right)u_j' X_1 + \sum_{j,\,k=1}^{2}\frac{1}{\sqrt{g_{11}}}\Gamma^k_{1j}\,u_j'\,X_k \\
&= -\sum_{j=1}^{2}\frac{g_{11,j}\,u_j'}{2g_{11}^{3/2}}X_1 + \sum_{j=1}^{2}\frac{1}{\sqrt{g_{11}}}\Gamma^1_{1j}\,u_j'\,X_1 + \sum_{j=1}^{2}\frac{1}{\sqrt{g_{11}}}\Gamma^2_{1j}\,u_j'\,X_2 \\
&= -\sum_{j=1}^{2}\frac{g_{11,j}\,u_j'}{2g_{11}^{3/2}}X_1 + \sum_{j=1}^{2}\frac{g_{11,j}\,u_j'}{2g_{11}^{3/2}}\,X_1 + \sum_{j=1}^{2}\frac{-g_{11,2}\,u_1' + g_{22,1}\,u_2'}{2\sqrt{g_{11}\,g_{22}}}\,e_2 \\
&= \sum_{j=1}^{2}\frac{-g_{11,2}\,u_1' + g_{22,1}\,u_2'}{2\sqrt{g_{11}\,g_{22}}}\,e_2
\end{aligned}$$

This implies that

$$\left[\frac{Dw}{dt}\right] = \left[\frac{D(X_1/\sqrt{g_{11}})}{dt}\right] + \frac{d\varphi}{dt} = \sum_{j=1}^{2}\frac{-g_{11,2}\,u_1' + g_{22,1}\,u_2'}{2\sqrt{g}} + \frac{d\varphi}{dt}.$$

(c) **Proposition (Liouville)** If $\alpha(s)$ is a parametrization by arc length of a neighborhood of a point $p \in S$ of a regular oriented curve C on an oriented surface S, and if $X(u_1, u_2)$ is an orthogonal parametrization of S in p and $\varphi(s)$ is the angle that X_1 makes with $\alpha'(s)$ in the given orientation, then

$$k_g = (k_g)_1 \cos\varphi + (k_g)_2 \sin\varphi + \frac{d\varphi}{ds},$$

where $(k_g)_1$ and $(k_g)_2$ are the geodesic curvatures of the coordinate curves $u_2 = \text{const.}$ and $u_1 = \text{const.}$ respectively.

Proof By setting $w = \alpha'(s)$ in the preceding Proposition, we obtain

$$k_g = \left[\frac{Dw}{ds}\right] = \frac{1}{2\sqrt{g}}\,(g_{22,1}\,u_2' - g_{11,2}\,u_1') + \frac{d\varphi}{ds}.$$

Let $\gamma_1(s) = \{X(u_1(s), u_2(s)) \mid u_1 = u_1(s),\ u_2 = \text{constant}\}$ and $\gamma_2(s) = \{X(u_1(s), u_2(s)) \mid u_2 = u_2(s),\ u_1 = \text{constant}\}$ be arc length parametrized coordinate curves with geodesic curvatures $(k_g)_1$ and $(k_g)_2$, respectively. Since

$$1 = \langle \gamma_i'(s), \gamma_i'(s)\rangle = g_{ii}(u_i')^2 \implies u_i' = \frac{1}{\sqrt{g_{ii}}}, \quad i = 1, 2,$$

and $\angle(X_1, \gamma_1') = 0$, $\angle(X_1, \gamma_2') = \dfrac{\pi}{2}$ for all s,

$$(k_g)_1 = \left[\frac{D\gamma_1'}{ds}\right] = \frac{1}{2\sqrt{g}}\,(-g_{11,2}\,u_1') = -\frac{g_{11,2}}{2g_{11}\sqrt{g_{22}}}, \quad (k_g)_2 = \left[\frac{D\gamma_2'}{ds}\right] = \frac{g_{22,1}\,u_2'}{2\sqrt{g}} = \frac{g_{22,1}}{2g_{22}\sqrt{g_{11}}}.$$

Also since $\varphi(s)$ is the angle that X_1 makes with $\alpha'(s) = u_1'X_1 + u_2'X_2$ in the given orientation, and since

$$\cos\varphi = \langle \alpha', \frac{X_1}{\sqrt{g_{11}}}\rangle = \sqrt{g_{11}}u_1' \quad \text{and} \quad \sin\varphi = \langle \alpha', \frac{X_2}{\sqrt{g_{22}}}\rangle = \sqrt{g_{22}}u_2',$$

we have

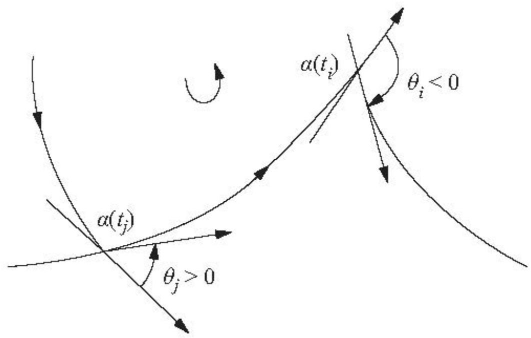
$$(k_g)_1 \cos\varphi + (k_g)_2 \sin\varphi + \frac{d\varphi}{ds} = \frac{1}{2\sqrt{g}}\,(g_{22,1}\,u_2' - g_{11,2}\,u_1') + \frac{d\varphi}{ds} = k_g.$$

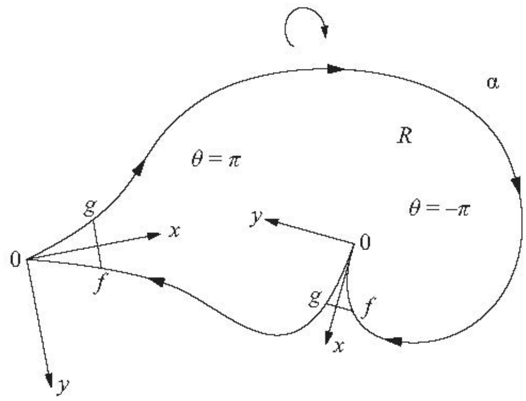
The Gauss-Bonnet Theorem and Its Applications

Definition Let $\alpha; [0, \ell] \to S$ be a continuous map from the closed interval $[0, \ell]$ into the regular surface S. We say that α is a simple, closed, piecewise regular, parametrized curve if

1. $\alpha(0) = \alpha(\ell)$,
2. $t_1 \neq t_2$, t_1, $t_2 \in [0, \ell)$, then $\alpha(t_1) \neq \alpha(t_2)$,
3. there exists a subdivision $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = \ell$ of $(0, \ell]$ such that α is differentiable and regular in each (t_i, t_{i+1}), $i = 0, \ldots, k$,
4. $\lim\limits_{t \to t_i^{\pm}} \alpha'(t) = \alpha'(t_i^{\pm}) \neq 0$ exists for each $i = 0, \ldots, k$.

The points $\alpha(t_i)$, $i = 0, \ldots, k$, are called the vertices of α and the traces $\alpha([t_i, t_{i+1}])$ are called the regular arcs of α. It is usual to call the trace $\alpha([0, \ell])$ of α, a closed piecewise regular curve.





The sign of the external angle in the case of a cusp.

Assume that S is oriented and let $|\theta_i|$, $0 \le |\theta_i| < \pi$, be the smallest determination of the angle from $\alpha'(t_i^-)$ to $\alpha'(t_i^+)$. If $|\theta_i| \ne \pi$, we give θ_i the sign of the determinant $(\alpha'(t_i^-),\, \alpha'(t_i^+),\, N)$. This means that if the vertex $\alpha(t_i)$ is not a "cusp", the sign of θ_i is given by the orientation of S. The signed angle θ_i, $-\pi < \theta_i < \pi$, is called the exterior angle at the vertex $\alpha(t_i)$.

Gauss-Bonnet Theorem (Local) Let $X : U \to S$ be an isothermal parametrization (i.e., $F = g_{12} = g_{21} = 0$, $E = g_{11} = g_{22} = G = \lambda^2(u_1, u_2)$) of an oriented surface S, where $U \subset \mathbb{R}^2$ is homeomorphic to an open disk and X is compatible with the orientation of S.

Let $R \subset X(U)$ be a simple region of S and let $\alpha : I \to S$ be such that $\partial R = \alpha(I)$.

- Assume that α is positively oriented, parametrized by arc length s, and
- let $\alpha(s_0), \ldots, \alpha(s_k)$ and $\theta_0, \ldots, \theta_k$ be, respectively, the vertices and the exterior angles of α.

Then

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s)\, ds + \iint_R K\, d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi, \qquad (*)$$

where $k_g(s)$ is the geodesic curvature of the regular arc of α and K is the Gaussian curvature of S.

Remarks

(a) If $R \subset X(U)$ is a geodesic triangle in S (that is, a triangle whose sides are arcs of geodesics), then $k_g(s) = 0$, $s \in (t_i, t_{i+1})$, $i = 1, 2, 3$ and

$$\iint_R K\, d\sigma = 2\pi - \sum_{i=1}^{3} \theta_i = 2\pi - \sum_{i=1}^{3} (\pi - \varphi_i) = \sum_{i=1}^{3} \varphi_i - \pi,$$

where $\varphi_i = \pi - \theta_i$ is defined to be the interior angle at the vertex $\alpha(t_i)$. In particular, if $K =$ constant, then

$$\sum_{i=1}^{3} \varphi_i - \pi = \iint_R K\, d\sigma = K\, A(R), \quad \text{where } A(R) = \text{ the area of } R.$$

(b) The restriction that the region R be contained in the image set of an isothermal parametrization is needed only to simplify the proof and to be able to use the theorem of turning tangents. As we shall see later (Corollary 1 of the global Gauss-Bonnet theorem) the above result still holds for any simple region of a regular surface. This is quite plausible, since Eq. $(*)$ does not involve in any way a particular parametrization.

Proof Let $u_1 = u_1(s)$, $u_2 = u_2(s)$ be the expression of X. By using a preceding Proposition, we have

$$k_g(s) = \frac{1}{2\sqrt{g}}\left(g_{22,1}\, u_2' - g_{11,2}\, u_1'\right) + \frac{d\varphi_i}{ds},$$

where $\varphi_i = \varphi_i(s)$ is a differentiable function which measures the positive angle from X_{u_1} to $\alpha'(s)$ in $[s_i, s_{i+1}]$. By integrating the above expression in every interval $[s_i, s_{i+1}]$ and adding up the results,

$$\begin{aligned}
&\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} k_g(s)\,ds \\
= &\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} \left(\frac{g_{22,1}}{2\sqrt{g}}\frac{du_2}{ds} - \frac{g_{11,2}}{2\sqrt{g}}\frac{du_1}{ds}\right) ds + \sum_{i=0}^{k}\left(\varphi_i(s_{i+1}) - \varphi_i(s_i)\right) \\
= &\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} \left(\frac{g_{22,1}}{2\sqrt{g}}\frac{du_2}{ds} - \frac{g_{11,2}}{2\sqrt{g}}\frac{du_1}{ds}\right) ds + \sum_{i=0}^{k-1}\left(\varphi_i(s_{i+1}) - \varphi_{i+1}(s_{i+1})\right) + \varphi_k(s_{k+1}) - \varphi_0(s_0) \\
= &\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} \left(-\frac{g_{11,2}}{2\sqrt{g}}\right) du_1 + \left(\frac{g_{22,1}}{2\sqrt{g}}\right) du_2 \pm 2\pi - \sum_{i=0}^{k}\theta_i \text{ since } \alpha \text{ is simple closed} \\
= &\iint_{X^{-1}(R)} \left[\left(\frac{g_{11,2}}{2\sqrt{g}}\right)_{u_2} + \left(\frac{g_{22,1}}{2\sqrt{g}}\right)_{u_1}\right] du_1\,du_2 + 2\pi - \sum_{i=0}^{k}\theta_i \\
&\text{since } \alpha \text{ is positively oriented and by the Green's Theorem} \\
= &\iint_{X^{-1}(R)} \left[\left(\frac{\lambda_{u_2}}{\lambda}\right)_{u_2} + \left(\frac{\lambda_{u_1}}{\lambda}\right)_{u_1}\right] du_1\,du_2 + 2\pi - \sum_{i=0}^{k}\theta_i \text{ since } g_{11} = g_{22} = \lambda^2 = g \\
= &\iint_{X^{-1}(R)} \left[(\log\lambda)_{u_2u_2} + (\log\lambda)_{u_1u_1}\right] du_1\,du_2 + 2\pi - \sum_{i=0}^{k}\theta_i \\
= &\iint_{X^{-1}(R)} \frac{\lambda^2}{2\lambda^2}\left(\Delta\log\lambda^2\right) du_1\,du_2 + 2\pi - \sum_{i=0}^{k}\theta_i \\
= &-\iint_{X^{-1}(R)} K\,\lambda^2\,du_1\,du_2 + 2\pi - \sum_{i=0}^{k}\theta_i \text{ by the Exercise 2 of 4-3} \\
= &-\iint_{R} K\,d\sigma + 2\pi - \sum_{i=0}^{k}\theta_i \text{ since } g = \lambda^2
\end{aligned}$$

Hence we have

$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} k_g(s)\,ds + \iint_R K\,d\sigma + \sum_{i=0}^{k}\theta_i = 2\pi.$$

Remark

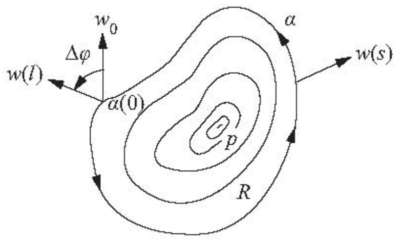
- Let $X : U \to S$ be an isothermal parametrization at a point $p \in S$, and let $R \subset X(U)$ be a simple region without vertices, containing p in its interior.
- Let $\alpha : [0, \ell] \to X(U)$ be a curve parametrized by arc length s such that $\partial R = \alpha([0, \ell])$.
- Let w_0 be a unit vector tangent to S at $\alpha(0)$ and let $w(s)$, $s \in [0, \ell]$, be the parallel transport of w_0 along α.

Then

$$\begin{aligned} 0 &= \int_0^\ell \left[\frac{Dw}{ds}\right] ds \quad \text{since } w \text{ is parallel along } \alpha \\ &\overset{\text{by } (\dagger)}{=} \int_0^\ell \frac{1}{2\sqrt{EG}}\left(G_{u_1}\frac{du_2}{ds} - E_{u_2}\frac{du_1}{ds}\right) ds + \int_0^\ell \frac{d\varphi}{ds}\, ds \\ &\overset{\text{by } (*)}{=} -\iint_R K\, d\sigma + \varphi(\ell) - \varphi(0). \end{aligned}$$

where $\varphi = \varphi(s)$ is a differentiable determination of the angle from X_{u_1} to $w(s)$. It follows that $\varphi(\ell) - \varphi(0) = \Delta\varphi$ is given by

$$\Delta\varphi = \iint_R K\, d\sigma.$$



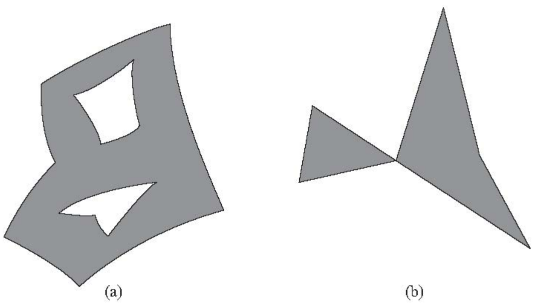
Now, $\Delta\varphi$ does not depend on the choice of w_0, and it follows from the expression above that $\Delta\varphi$ does not depend on the choice of $\alpha(0)$ either. By taking the limit

$$\lim_{R\to p} \frac{\Delta\varphi}{A(R)} = K(p),$$

where $A(R)$ denotes the area of the region R, we obtain the desired interpretation of K.

Definitions

- Let S be a regular surface. A connected region $R \subset S$ is said to be regular if R is compact and its boundary ∂R is the finite union of (simple) closed piecewise regular curves which do not intersect (the region in (a) is regular, but that in (b) is not).



- A simple region which has only three vertices with exterior angles $\alpha_i \neq 0$, $i = 1, 2, 3$, is called a triangle.
- A triangulation of a regular region $R \subset S$ is a family $\mathscr{F}$ of triangles T_i, $i = 1, \ldots, n$, such that

1. $\displaystyle\bigcup_{i=1}^{n} T_i = \bigcup_{T_i \in \mathscr{F}} T_i = R$,
2. If $T_i \cap T_j \neq \emptyset$, then $T_i \cap T_j$ is either a common edge of T_i and T_j or a common vertex of T_i and T_j.

- Given a triangulation $\mathscr{F}$ of a regular region $R \subset S$ of a surface S, we shall denote by
  - F the number of triangles (faces),
  - E the number of sides (edges),
  - V the number of vertices of the triangulation.

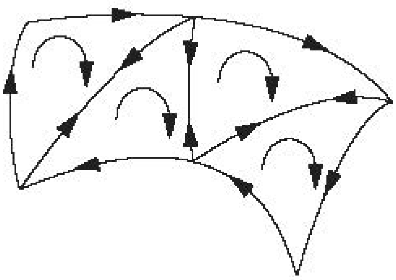
  The number

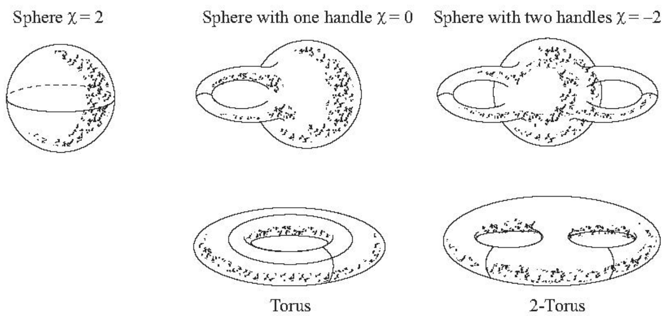
  $$F - E + V = \chi$$

  is called the Euler-Poincaré characteristic of the triangulation.

Propositions (without proofs)

1. Every regular region of a regular surface admits a triangulation.
2. Let S be an oriented surface and $\{X_\alpha\}$, $\alpha \in A$, a family of parametrizations compatible with the orientation of S. Let $R \subset S$ be a regular region of S. Then there is a triangulation $\mathscr{F}$ of R such that every triangle $T \in \mathscr{F}$ is contained in some coordinate neighborhood of the family $\{X_\alpha\}$. Furthermore, if the boundary of every triangle of $\mathscr{F}$ is positively oriented, adjacent triangles determine opposite orientations in the common edge.





3. If $R \subset S$ is a regular region of a surface S, the Euler-Poincaré charateristc does not depend on the triangulation of R. It is convenient, therefore, to denote it by $\chi(R)$.
4. Let $S \subset \mathbb{R}^3$ be a compact connected surface; then one of the values $2, 0, -2, \ldots, -2n, \ldots$ is assumed by the Euler-Poincaré charateristic $\chi(S)$. Furthermore, if $S' \subset \mathbb{R}^3$ is another compact surface and $\chi(S) = \chi(S')$, then S is homeomorphic to S'.

   In other words, every compact connected surface $S \subset \mathbb{R}^3$ is homeomorphic to a sphere with a certain number g of handles. The number

   $$g = \frac{2 - \chi(S)}{2} \quad \text{is called the genus of } S.$$

4. Let $R \subset S$ be a regular region of an oriented surface S and let $\mathscr{F}$ be a triangulation of R such that every triangle $T_j \in \mathscr{F}$, $j = 1, \ldots, k$, is contained in a coordinate neighborhood $X_j(U_j)$ of a family of parametrizations $\{X_\alpha\}$, $\alpha \in A$, compatible with the orientation of S. Let f be a differentiable function on S. Then the sum

   $$\sum_{j=1}^{k} \iint_{X_j^{-1}(T_j)} f(u_j, v_j) \sqrt{E_j G_j - F_j^2}\, du_j\, dv_j$$

does not depend on the triangulation $\mathscr{F}$ or on the family $\{X_j\}$ of parametrizations of S. This sum has, therefore, a geometrical meaning and is called the integral of f over the regular region R. It is usually denoted by

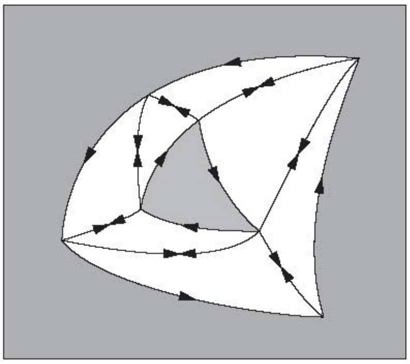
$$\iint_R f\, d\sigma.$$

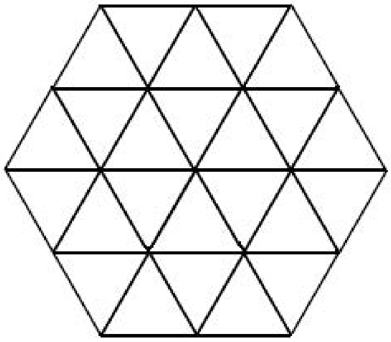
Gauss Bonnet Theorem (Global) Let $R \subset S$ be a regular region of an oriented surface and let $C_1, \ldots, C_n$ be the closed, simple, piecewise regular curves which form the boundary ∂R of R. Suppose that each C_i is positively oriented and let $\theta_1, \ldots, \theta_p$ be the set of all exterior angles of the curves $C_1, \ldots, C_n$. Then

$$\sum_{i=1}^{n} \int_{C_i} k_g(s)\, ds + \iint_R K\, d\sigma + \sum_{\ell=1}^{p} \theta_\ell = 2\pi\chi(R) \quad (**),$$

where s denotes the arc length of C_i, and the integral over C_i means the sum of integrals in every regular arc of C_i.

Proof Consider a triangulation $\mathscr{F} = \{T_j\}_{j=1}^{F}$ of the region R such that every triangle T_j is contained in a coordinate neighborhood of a family of isothermal parametrizations compatible with the orientation of S. Such a triangulation exists by a preceding Proposition. Furthermore, if the boundary of every triangle of $\mathscr{F}$ is positively oriented, we obtain opposite orientations in the edges which are common to adjacent triangles.





By applying to every triangle the local Gauss-Bonnet theorem and adding up the results we obtain, using a preceding Proposition and the fact that each interior side is described twice in opposite orientations,

$$\sum_i \int_{C_i} k_g(s)\, ds + \iint_R K\, d\sigma + \sum_{j=1}^{F}\sum_{k=1}^{3} \theta_{jk} = 2\pi F,$$

where F denotes the number of triangles of $\mathscr{F}$, and θ_{j1}, θ_{j2}, θ_{j3} are the exterior angles of the triangle T_j.

We shall now introduce the interior angles of the triangle T_j, given by $\varphi_{jk} = \pi - \theta_{jk}$, $k = 1, 2, 3$. Thus,

$$\sum_{j=1}^{F}\sum_{k=1}^{3} \theta_{jk} = \sum_{j=1}^{F}\sum_{k=1}^{3} (\pi - \varphi_{jk}) = \sum_{j=1}^{F}\sum_{k=1}^{3} \pi - \sum_{j=1}^{F}\sum_{k=1}^{3} \varphi_{jk} = 3\pi F - \sum_{j=1}^{F}\sum_{k=1}^{3} \varphi_{jk}.$$

We shall use the following notation:

$$\begin{aligned} E_e &= \text{number of exterior edges of } \mathscr{F}, \\ E_i &= \text{number of interior edges of } \mathscr{F}, \\ V_e &= \text{number of exterior vertices of } \mathscr{F}, \\ V_i &= \text{number of interior vertices of } \mathscr{F}. \end{aligned}$$

Since the curves C_i are closed, $E_e = V_e$. Furthermore, it is easy to show by induction that

$$3F = 2E_i + E_e$$

and therefore that

$$\sum_{j=1}^{F}\sum_{k=1}^{3}\theta_{jk} = 2\pi E_i + \pi E_e - \sum_{j=1}^{F}\sum_{k=1}^{3}\varphi_{jk}.$$

Observe that

- the exterior vertices may be either vertices of some curve C_i or vertices introduced by the triangulation, and if V_{ec} is the number of vertices of the curves C_i and V_{et} is the number of exterior vertices of the triangulation which are not vertices of some curve C_i, then $V_e = V_{ec} + V_{et}$.
- the sum of angles around each interior vertex is 2π, the sum of angles around each exterior vertex of the triangulation is π.

We obtain

$$\sum_{j=1}^{F}\sum_{k=1}^{3}\theta_{jk} = 2\pi E_i + \pi E_e - 2\pi V_i - \pi V_{et} - \sum_{\ell=1}^{p}(\pi - \theta_\ell).$$

By adding πE_e to and subtracting it from the expression above and taking into consideration that $E_e = V_e$, we conclude that

$$\sum_{j=1}^{F}\sum_{k=1}^{3}\theta_{jk} = 2\pi E_i + 2\pi E_e - 2\pi V_i - \pi V_e - \pi V_{et} - \pi V_{ec} + \sum_{\ell=1}^{p}\theta_\ell = 2\pi E - 2\pi V + \sum_{\ell=1}^{p}\theta_\ell.$$

By putting things together, we finally obtain

$$\sum_{i=1}^{n}\int_{C_i} k_g(s)\,ds + \iint_R K\,d\sigma + \sum_{\ell=1}^{p}\theta_\ell = 2\pi(F - E + V) = 2\pi\chi(R).$$

Corollary If R is a simple region of S, then

$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} k_g(s)\,ds + \iint_R K\,d\sigma + \sum_{i=0}^{k}\theta_i = 2\pi$$

Corollary Let S be an orientable compact surface; then

$$\iint_S K\,d\sigma = 2\pi\chi(S)$$